Quantum correlations II

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- The purpose of this lecture is to review some of the standard facts from probability theory as well as to present the general form of two point (classical) correlation function.

- We begin with basic definitions:

- σ-algebra:

Definition 1. *Let Ω be a set. $\mathcal{F}$ is a σ-algebra if $\mathcal{F} \subset 2^{\Omega}$ and*

1. $\Omega \in \mathcal{F}$,
2. *if* $A \in \mathcal{F}$ *then* $\Omega \setminus A \in \mathcal{F}$,
3. *if* $A_i \in \mathcal{F}$ *for* $i = 1, 2 \ldots$, *then* $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measure space.

where 2^{Ω} denotes the family of all subsets of Ω.

- **Definition 2.** *A probability measure p on $(\Omega, \mathcal{F})$ is a function $p : \mathcal{F} \to [0,1]$ such that*

  1. $p(\Omega) = 1$
  2. $p(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} p(A_i)$ *if* $A_i \in \mathcal{F}$ *for* $i = 1, 2 \ldots,$ *and* $A_i \cap A_j = \emptyset$ *for* $i \neq j$.

- and finally the basic concept of this lecture:

- **Definition 3.** *A probability space is a triple $(\Omega, \mathcal{F}, p)$ where Ω is a space (sample space), $\mathcal{F}$ is a σ-algebra (a family of events), and p is a probability measure on $(\Omega, \mathcal{F})$.*

- **Remark 4.** *In probability theory, an "elementary event" is considered as the principal undefined term. Intuitively speaking, an elementary event has the meaning of the possible outcome of some physical experiment.*

- To illustrate these definitions we give:

- **Example 5.** *Discrete probability space.*
Let Ω be a countable (finite) set. Let us take $\mathcal{F} = 2^{\Omega}$. We put, for $A \subset \Omega$, $p(A) = \sum_{\omega \in A} q(\omega)$, where $q(\omega) \in [0,1]$, $\omega \in \Omega$, are numbers such that $\sum_{\omega \in \Omega} q(\omega) = 1$.
$(\Omega, \mathcal{F}, p)$ is a probability space.

- The last statement can be easily checked.

- Our next example is:

- **Example 6.** *Continuous probability space.*
  To be specific we put $\Omega = [0,1] \subset \mathbb{R}$, $\mathcal{F}$ *to be a* σ*-algebra of Borel sets* $\mathcal{B}$ *in* $[0,1]$, *and* λ *to be the Lebesgue measure.*
  $(\Omega, \mathcal{F}, \lambda)$ *is a probability space.*

- Borel sets are defined as the σ-algebra generated by open sets.

- Again, it is an easy excercise to check the above statement.

- The notion of stochastic variable can be considered as a counterpart of a (classical) observable.

- **Definition 7.** *Let* $(\Omega, \mathcal{F}, p)$ *be a probability space. A measurable, real valued function* $f : \Omega \to \mathbb{R}$ *is called a stochastic variable.*

- To speak about *correlations*, it is convenient to begin with the opposite notion - with the notion of *independence*.

- **Definition 8.** *Let $(\Omega, \mathcal{F}, p)$ be a probability space. We say that two events $A, B \in \mathcal{F}$ are* ***independent*** *if*

$$p(A \cap B) = p(A)p(B) \tag{1}$$

More generally, the events A_i, $i = 1, ..., n$ in $\mathcal{F}$ are independent if

$$p(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = \prod_{j=1}^{k} p(A_{i_j}) \tag{2}$$

for every $k = 1, ..., n$ and $i_1, ..., i_k$ such that $1 \leq i_1 < i_2, ..., i_k \leq n$.

- The concept of independence can be extended for stochastic variables:

- We say that stochastic variables $X_1,\ X_2,\ \ldots, X_n$ are independent if and only if events $(X_i \in B_i)$, $i = 1, ..., n$, are independent where

$$(X_i \in B_i) \equiv \{\omega : X_i(\omega) \in B_i\} \tag{3}$$

  for arbitrary Borel sets $B_i \subset \mathbb{R}$.

- We define an expectation value $E(X)$ of a stochastic variable X as

$$E(X) = \int X dp \equiv \int X(\omega) dp(\omega) \tag{4}$$

- We interpret $E(X)$ as the mean value of the stochastic variable X provided that the probability of events is given by the probabilistic measure p.

- **Definition 9.** *Let two stochastic variables X, Y be given. Moreover, let their second moments be finite, i.e. $E(X^2)$, $E(Y^2) < \infty$. We say that two stochastic variables X and Y are* $\mathbf{uncorrelated}$ *if*

$$E(XY) = E(X)E(Y) \tag{5}$$

- It is easy an easy excercise to check that:

$$(E(XY) - E(X)E(Y))^2 \leq (E(X^2) - E(X)^2)(E(Y^2) - E(Y)^2) \tag{6}$$

- The following definition should be clear, $Halmos$:

- **Definition 10.** *A **correlation coefficient** $C(X,Y)$ is defined as*

$$C(X,Y) = \frac{E(XY) - E(X)E(Y)}{(E(X^2) - E(X)^2)^{\frac{1}{2}}(E(Y^2) - E(Y)^2)^{\frac{1}{2}}} \tag{7}$$

- Taking into account inequality (6), it is obvious that $C(X,Y) \in [-1,1]$.

- If $C(X,Y)$ is equal to 0 then stochastic variables X and Y are $uncorrelated$.

- Further, if $C(X,Y) \in (0,1]$, then X, Y are said to be $correlated$ (positive correlated).

- Finally when $C(X,Y) \in [-1,0)$, stochastic variables X and Y are said to be $anti\ correlated$ (negative correlated).

- The above introduced notions for stochastic variables, independence and uncorrelatedness, are related to each other. Namely, $Halmos$:

- **Proposition 11.** *$(\Omega = [0,1],\ \mathcal{F},\ p)$ be a probability space. Assume that stochastic variables are independent and integrable. Then they are also uncorrelated.*

- However, these two notions are not equivalent. To illustrate, $Halmos$:

- **Example 12.** *Let $(\Omega = [0,1],\ \mathcal{F},\ \lambda)$ be the probability space given in Example 6. Define stochastic variables f and g as $f(x) = \sin 2\pi x$ and $g(x) = \cos 2\pi x$. Then the expectation value of $f \cdot g$ is given by $E(fg) = \int_0^1 \sin 2\pi x \cos 2\pi y dx = 0$. Hence , they are uncorrelated. On the other hand, let us define $D = [0, \epsilon)$ where ϵ is sufficiently small. $\lambda(\{x : \sin 2\pi x \in D\} \cap \{x : \cos 2\pi x \in D\}) = 0$ while $\lambda(\{x : \sin 2\pi x \in D\}) \neq 0$ and $\lambda(\{x : \cos 2\pi x \in D\}) \neq 0$. Therefore f and g are not independent (but they are uncorrelated!).*

- Relation between a measure and a state (*this will be essential for a quantization of probability*):

- Let E be a locally compact Hausdorff space.

- A positive Radon measure is a positive linear map $\phi : C_{\mathfrak{K}}(E) \to \mathbb{R}$ where $C_{\mathfrak{K}}(E)$ denotes the set of continuous functions with compact support, and ϕ positive means $\phi(f) \geq 0$ for any $f \geq 0$.

- A Borel measure is a measure defined on the σ-algebra generated by compact subsets of E such that the measure of every compact subset is finite.

- Let E be a locally compact space, μ be a positive (Borel) measure on E and f be a continuous function on E with a compact support.

- Consider the following map:

$$C_{\mathfrak{K}}(E) \ni f \mapsto \int f d\mu \in \mathbb{C}. \tag{8}$$

- Verify, that

1. $\int (f+g)d\mu = \int f d\mu + \int g d\mu$
2. $\int cf d\mu = c \int f d\mu$ for $c \in \mathbb{C}$
3. $\int f d\mu \geq 0$ for $f \geq 0$

- In (8), each positive measure on E defines a positive linear form on $C_{\mathfrak{K}}(E)$.

- The converse implication is given by the Markov-Riesz-Kakutani theorem.

Theorem 13. *If φ is a linear, positive, continuous form on $C_{\Re}(E)$ then there exist a unique positive Borel measure μ on E such that*

$$\varphi(f) = \int_E f d\mu \qquad f \in C_{\Re}(E). \tag{9}$$

If additionally E is compact and $\varphi(\mathbb{1}) = 1$ then μ is a probability measure. Consequently, normalized forms and probability measures are in $1-1$ correspondence.

- Theorem 13 establishes a one-to-one correspondence between the positive Borel measures (in fact, regular positive Borel measures) and the Radon measures such that $\mu(E) = \|\mu\|$.

- The standard integral notation $\mu(f) = \int_E f d\mu$, $f \in C_{\mathfrak{R}}(E)$ implicitly identifies these concepts. Thus, in in sequel, *the term measure will be used interchangeably to denote them.*

- Assume additionally, temporary, that E is a compact space. $C_{\mathfrak{R}}(E) \equiv C(E)$ equipped with the supremum norm $\|f\| = \sup_{x \in E} |f(x)|$, where $f \in C(E)$, is a Banach space and the map (8) is continuous one (if $C(E)$ is considered as a Banach space).

- One can say even more: $C(E)$ can be furnished with an algebraic structure turning $C(E)$ into an abelian C^*-algebra with unit (in next lectures we will come back to this point!).

- Calling normalized forms (8) as states one gets a *one-to-one correspondence between a probability measure on E and the corresponding state*!

- Some more notation: we wish to explain the term $Riemann\ approximation\ property$ which was used in the first lecture.

- For a Borel measure μ on a locally compact Hausdorff space E and a function $f \in C_{\mathfrak{K}}(E)$ we denote $\mu(f) = \int_E f d\mu$ (cf Theorem 13). Denote by $\mathfrak{M}(E)$ the collection of Radon measures on E. Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathfrak{M}(E)$.

- We say that the net $\{\mu_n\}$ is weakly convergent to μ if $\mu_n(f) \to \mu(f)$ for any function $f \in C_{\mathfrak{K}}(E)$ (the vague topology)

- We will need the notion of Dirac's (point) measure δ_a, where $a \in E$.

- Such measures are determined by the condition:

$$\delta_a(f) = f(a) \tag{10}$$

- Finally, we say that a measure μ has a finite support if it can be written as a linear (finite) combination of δ_a's.

- *see Bourbaki*

  Theorem 14. *Any positive finite measure μ on E is a limit point, in the vague topology, of a convex hull of positive measures having a finite support contained in the support of μ.*

- **Remark 15.** *1. This result will be not valid in the non-commutative setting. It is true only in the classical theory!*

- To describe (classical) composite systems we need the concept of product strucrures.

- Let $(\Omega_i, \mathcal{F}_i, p_i)$, $i = 1, 2$ be a probability space.

- The product of two probability spaces (which is also a probability space) is the Cartesian product

$$(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, p_1 \times p_2),$$

where the product measure $p_1 \times p_2$ is defined as

$$p_1 \times p_2(A \times B) = p_1(A)p_2(B)$$

for all $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. $\mathcal{F}_1 \times \mathcal{F}_2$ denotes the σ-algebra generated by sets of the form $\{A \times B; A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$.

- Let χ_Y be an indicator function of a measurable set Y.

- Assume that for any $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$ functions $\chi_{A\times\Omega_2}$ and $\chi_{\Omega_1\times B}$ are uncorrelated.

- Then one has

$$\mu(\chi_{A\times\Omega_2} \cdot \ \chi_{\Omega_1\times B}) = \mu(\chi_{A\times\Omega_2})\mu(\chi_{\Omega_1\times B}) \tag{11}$$

- The measures $\mu_1(A) = \mu(\chi_{A\times\Omega_2})$, $\mu_2(B) = \mu(\chi_{\Omega_1\times B})$ are called the marginal measures.

- Consequently, if any two events give rise to uncorrelated indicator functions, for a measure μ on the product measurable (probability) space then $\mu = \mu_1 \times \mu_2$.

- *Classical composite systems.*

- In classical statistical mechanics, a system is described by its phase space Γ, a probability measure μ, and a one parameter family of measure preserving maps $T_t : \Gamma \to \Gamma$.

- The phase space Γ records the allowed collection of system coordinates and momenta. The measure μ is characterizing our knowledge about the system.

- Finally, the family of maps $\{T_t\}$, where either $t \in \mathbb{R}$ or $t \in \mathbb{R}^+$, is designed to describe a time evolution of the system.

- Suppose we have two such systems $(\Gamma_i,\, \mu_i,\, T_t^i)$, $i = 1, 2$.

- We wish to form one bigger system consisting of these two given subsystems - thus we want to form a *composite* system.

- BUT, one should realize that there are three different types of independence; see $Streater$:

- 1. logical independence,
  2. dynamical independence,
  3. statistical independence.

- Logical independence means that we are implicitly assuming that the values allowed for the first component (so for Γ_1) do not depend on the values taken by the second component (so those in Γ_2). This leads to the conclusion that the allowed values for the composite system are given by the Cartesian product of its components, i.e. $\Gamma = \Gamma_1 \times \Gamma_2$. From now on we make this assumption.

- The dynamical independence means that the global evolution is described by the product of maps, i.e. $T_t = T_t^1 \times T_t^2$. The important point to note here is that this independence excludes any interaction between the two subsystems. This kind of independence will not be assumed.

- The last one, the statistical independence means that the global measure describing our knowledge about the system is a product measure.

- It is a simple matter to check that this independence is related to that described in the first part of this lecture.

- We will not assume the statistical independence.

- We arrived at: *a composite system is characterized by the triple* $(\Gamma \equiv \Gamma_1 \times \Gamma_2,\ \mu,\ T_t)$, *where the probability measure* μ *is defined on the Cartesian product of two measurable spaces* $(\Gamma_1 \times \Gamma_2, \mathcal{F}_1 \times \mathcal{F}_2)$, *and finally,* T_t *is a global evolution defined on* Γ.

- Classical correlation functions.

- Let us assume that Γ_1, Γ_2 are compact sets and we consider only continuous stochastic variables.

- Hence, a global (classical) observable is given by a function $f \in C(\Gamma_1 \times \Gamma_2)$ while an observable associated with a subsystem is given by $f_i \in C(\Gamma_i)$, $i = 1, 2$ respectively.

- There is the identification:

$$C(\Gamma_1 \times \Gamma_2) = C(\Gamma_1) \otimes C(\Gamma_2) \tag{12}$$

where on the right hand side of (12) $\otimes$ stands for the tensor product, (more details on tensor products will be provided in a separate lecture!)

- Now we wish to consider a two point correlation function, where an observable f_i is associated with a subsystem i, $i = 1, 2$.

- Note, that quantum counterparts of such correlation functions are typical in Quantum Information Theory.

- To simplify the notation we will identify the function f_1 (defined on Γ_1) with the function $f_1 \otimes \mathbb{1}_{\Gamma_2}$ (defined on $\Gamma_1 \times \Gamma_2$); and analogously for f_2.

- Let $f_i \in C(\Gamma_i)$, $i = 1, 2$ and consider functionals $\varphi(\cdot)$

$$\varphi(f_1 \otimes f_2) = \varphi(f_1 f_2) \equiv \varphi_\mu(f_1 f_2) \equiv \int_{\Gamma_1 \times \Gamma_2} f_1(q_1) f_2(q_2) d\mu \tag{13}$$

- Taking into account the weak-* Riemann approximation property

$$\begin{aligned} \varphi(f_1 f_2) &= \lim_{n\to\infty} \int_{\Gamma_1 \times \Gamma_2} f_1(\gamma_1) f_2(\gamma_2) d\mu_n \\ &= \lim_{n\to\infty} \int_{\Gamma_1 \times \Gamma_2} f_1(\gamma_1) f_2(\gamma_2) (\sum_n \lambda_n d\delta^n_{(a_{1,n}, a_{2,n})}) \end{aligned} \tag{14}$$

where $\delta_{(a,b)}$ stands for the Dirac's measure supported by (a, b), $\lambda_n \geq 0$ and $\sum_n \lambda_n = 1$.

- for a point measure, one has

$$\delta_{(a,b)} = \delta_a \times \delta_b. \tag{15}$$

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$$\begin{aligned}\varphi(f_1 f_2) &= \lim_{n\to\infty} \sum_n \lambda_n \int_{\Gamma_1} f_1(\gamma_1) d\delta^n_{a_{1,n}}(\gamma_1) \int_{\Gamma_2} f_2(\gamma_2) d\delta^n_{a_{2,n}}(\gamma_2) \\ &= \lim_{n\to\infty} \sum_n \lambda_n \varphi_{\delta_{a_{1,n}}}(f_1) \varphi_{\delta_{a_{2,n}}}(f_2) \\ &= \lim_{n\to\infty} \sum_n \lambda_n (\varphi_{\delta_{a_{1,n}}} \otimes \varphi_{\delta_{a_{2,n}}})(f_1 \otimes f_2)\end{aligned} \tag{16}$$

for any $f_i \in C(\Gamma_i)$, $i = 1, 2$.

- Consequently

$$\varphi_\mu(f_1 \otimes f_2) = \lim_{n\to\infty} \sum_n \lambda_n (\varphi_{\delta_{a_{1,n}}} \otimes \varphi_{\delta_{a_{2,n}}})(f_1 \otimes f_2) \qquad (17)$$

  for any $f_i \in C(\Gamma_i)$, $i = 1, 2$.

- **Corollary 16.** *For a classical case, any two point correlation function of bipartite system is the limit of a convex combination of product states.*

- This means that it is of very specific (separable form). We will see that it is not true for the quantum case.

- The important point to note here is that two point correlation function is the principal ingredient of correlation coefficient.